Propositional Stability

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1. Introduction

This short article adumbrates a new and useful notion relevant to so-called combined modal logics[1], Markov Logic Networks, a variety of combined logics, and Transactional Logic (forthcoming). Specifically, we seek to define and identify the conditions under which truth-values remain stable when interacted with by ***more than one logic***.

"Under what conditions", we might ask, "do propositions remain unchanged in their truth determinations?" Furthermore, "how might we proceed to calculate that and track such changes?" ***Propositional Stability*** is introduced to that end.

More rigorously, the concept is specifically introduced to:

- Formally define phenomena already encountered in several existing areas of inquiry.
- Be useful in analyzing the burgeoning field of combined logics (modal or otherwise).
- Allow for a greater variety of perspectives including *proposition-focused perspectives* in addition to *logic-focused perspectives*[2].
- Link active areas of research today to new areas that will be introduced into the field of logic at some future time.

2. Justification and Motivation

Standard practice in logic and mathematics has it that introducing a new concept rarely requires much justification insofar as:

1. The new concept builds upon a pre-existing body of work, area of study, or existing tool.

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2. Expands the analytical toolkit used to understand a specific mathematical area.

Consider, for example, truth-tables (improving over the syllogism and its humble square), *Cohen Forcing* in Set Theory, Tarski's introduction of metalanguages, and so on. Still, I'd like to provide <u>explicit</u> justification for the concept contained herein.

The argument from practice (truth-assignments):

P1. It is widely accepted as a matter of practice in philosophy, logic, and computer science that propositions (or sentences if you prefer) are assigned a truth-value by way of a truth-assignment (interpretation function).

P2. Truth-assignments vary within the same logic. (e.g. - Propositions that are true under every interpretation are called tautologies).

C. Therefore, assigning propositions various truth-values <u>within the same logic</u> is already widely accepted as a matter of practice.

We review what is perhaps the most familiar definition of a truth-assignment as it defined within the semantics of Classical Logic.

Definition 1.0. *Classical Zero-Order Alphabet.*

1. *A* is a set of *propositional variables*. $A = \{A_0, A_1, \ldots, B_0, B_1, \ldots, \ldots, Z_0, Z_1, \ldots\}$.

2. Ω is the set of primitive *logical connectives* for L_1. $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$.
   a. Ω_0 is the set of *logical connectives* of *arity* 0. $\Omega_0 = \{\bot, \top\}$.
   b. Ω_1 is the set of *logical connectives* of *arity* 1. $\Omega_1 = \{\neg\}$.
   c. Ω_2 is the set of *logical connectives* of *arity* 2. $\Omega_2 = \{\rightarrow\}$.

3. The set $A \cup \Omega$ comprises the *alphabet* of L_1.

4. The *well-formed formulae* (*wff*) of L_1 are recursively defined as follows:
   a. Any δ, where δ is a sentential variable of L_1, is a formula.
   b. If δ is a formula then, $\neg\delta$ is a formula.
   c. If δ and φ are formulas then, $\delta \rightarrow \varphi$ is a formula.
   d. $\top$ and $\bot$ are formulas.
   e. There are no other wff.
   f. Comprises the *grammar* of L_1.
   g. Let *wff*(L_1) denote the set of all wff in L_1.

Definition 1.1. *Classical Zero-Order Truth Assignment.*

1. A triple $\langle V, \Phi, \Phi^* \rangle$ is an L^T structure just in case:

a. V is a theory.
b. $V = A(V) \cup B(V)$ such that:
   i. $A(V) \subseteq A$ and $A(V) \neq \emptyset$
   ii. $A(V) \subseteq B(V)$
   iii. $B(V) \subseteq wff(L^T)$

2. We call Φ a ***propositional interpretation function*** (for the non- concatenated *wff*) of L^T.
   a. $\Phi: A(V) \rightarrow \{\top, \perp\}$ such that:
      i. $\Phi(p) = \top$ else $\Phi(\mathrm{p}) = \perp$

3. We call Φ^* a ***sentential interpretation function*** (for the concatenated *wff*) of L^T – the procedure for constructing that Φ^* is explained below.
   a. $\Phi^*: B(V) \rightarrow \{\top, \perp\}$ such that:
      i. For all $p \in \mathrm{A(V)}$, $\Phi^*(p) = \Phi(p)$
      ii. $\Phi^*(p) = \top$ *just in case* $\Phi^*(p) \neq \perp$
      iii. $\Phi^*(\perp) = \perp$
      iv. $\Phi^*(\top) = \top$
      v. $\Phi^*(\neg p) = \top$ *just in case* $\Phi^*(p) = \perp$
      vi. $\Phi^*(p \rightarrow q) = \top$ *just in case* $\Phi^*(p) = \perp$ or $\Phi^*(q) = \top$
      vii. $\Phi^*(p \;\&\; q) = \top$ *just in case* $\Phi^*(p) = \top = \Phi^*(q)$
      viii. $\Phi^*(p \vee q) = \top$ *just in case* $\Phi^*(p) = \top$ or $\Phi^*(q) = \top$
      ix. $\Phi^*(\mathrm{p} \leftrightarrow \mathrm{q}) = \top$ *just in case* $\Phi^*(p) = \Phi^*(q)$
   b. If $\Phi^*(p) = \top$, then $\Phi^* \models p$
   c. For all $p \in \mathrm{V}$, if $\Phi^* \models p$, then Φ^* is a model of V

The argument from practice (modal logic):

P1. Combined modal logics have been studied in detail (and continue to serve as an area of fruitful research).

P2. Propositions are evaluated according to fragments, extensions, and combinations of the standard modal axioms (**D**, **T**, **B**, **S4**, **S5**).

C. Therefore, assigning propositions various truth-values within many modal axiom systems is already widely accepted as a matter of practice.

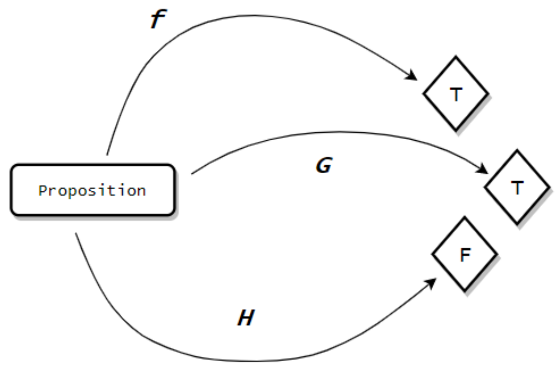

Fig. 1 - A single proposition under multiple interpretations or truth-assignments.

The aim here is to provide a robust set of concepts and definitions for them.

3. *Conventions*

Where:

1. ⊶ ∈ ℕ
2. ⋇ ∈ {*a*, …, *z*, …}
3. {*a*, …, *z*, …} = ℕ

We write (quotes[3] are dropped):

1. ML⊶⋇ to denote a semantics (model or truth-assignment M) for a language L⊶ with ⋇-many truth values.

2. VML⊶⋇(*p*) to denote a truth-evaluation of proposition *p* under semantics (model or truth assignment M) for a language L⊶ with ⋇-many truth-values.

3. VML1aVML2b(*p*)* to denote any possible truth-evaluation of *p* to a truth-value *t* in semantics ML2b such that: t ∈ ML2b and t ∉ ML1a.

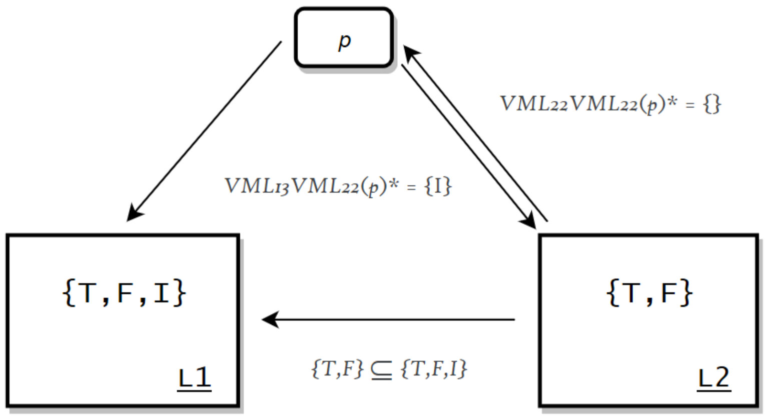

Fig. 2 – Simple depiction demonstrating truth-functional relationships between Boolean and Kleene 3-Value Logics.

4. Definitions

Definition 2.0. *Instruction set.*

An ***instruction set*** is a finite procedure or algorithm mapping one input to one output.

Definition 2.1. *Strong propositional stability.*

1. A proposition or sentence *p* evaluated under semantics ML1a will preserve its exact truth-value under semantics ML2b whenever:

    a. ML1a ⊆ ML2b; and
    b. no ***instruction set*** exists to map VML1a(*p*) to any other truth-value.

    p is then said to exhibit ***strong propositional stability***.

2. A proposition *p* exhibits ***strong propositional stability*** when and only when:

    a. VML1a(*p*) = VML2b(*p*)
    b. t ∈ VML1a ∪ VML2b
    c. VML1a(*p*) ∪ *t*
    d. No instruction set exists to map VML1a(*p*) to *t*

Definition 2.2. *Weak propositional stability.*

1. A proposition or sentence *p* evaluated under semantics ML1a will preserve its range of truth-values under semantics ML2b whenever $a \subseteq b$ and no ***instruction set*** exists to

map VML1a(p) to any VML1aVML2b(p)*. p is then said to exhibit ***weak propositional stability***.

2. A proposition p exhibits ***weak propositional stability*** when and only when:

   a. VML1a(p) $\subset$ VML2b(p)
   b. No ***instruction set*** exists to map VML1a(p) to any VML1aVML2b(p)*.

Definition 2.3. *Truth stability.*

1. A proposition or sentence p evaluated under semantics ML1a will preserve its exact truth-value under semantics ML2b whenever $a \subseteq b$. p is then said to exhibit ***truth stability***.

2. A proposition p exhibits ***truth stability*** when and only when VML1a(p) = VML2b(p).

Definition 2.4. *Propositional instability.*

A proposition p exhibits ***propositional instability*** whenever it does not exhibit ***weak propositional stability***.

Definition 2.5. *Truth instability.*

A proposition p exhibits ***truth instability*** whenever it does not exhibit ***truth stability***.

5. *Discussion*

Remark 1. ***Strong propositional stability*** entails ***weak propositional stability*** and ***truth stability***.

Discussion: ***Strong propositional stability*** requires that a proposition retains its exact truth-value under two logics and that no method exists for that truth-value to vary. Thus, it is constrained by the same range of truth-values.

Remark 2. ***Truth stability*** guarantees only incidental sameness of truth-assignment.

Discussion: In some cases, ***truth stability*** will converge with ***strong propositional stability***, in others it will not.

Remark 3. Every proposition will exhibit ***strong propositional stability*** when evaluated under the same logic (provided that the logic is ***eternalist*** – or remains invariant across time).

Remark 4. Tautologies under some logic L exhibit ***strong propositional stability*** within L.

6. *Results*

Fact 1. Any proposition truth-evaluated under a Boolean logic will exhibit ***strong propositional stability when*** truth-evaluated under a Kleene 3-Value Algebra.

> **Proof:** Obvious. No single proposition already assigned a truth-value of 'true' or 'false' can receive a truth-value of 'indeterminate' or 'true and false'. ■

Fact 2. Given:

> 1. Sound monotonic axiom systems $\Omega 1$, $\Omega 2$
> 2. $\Omega 1 \subset \Omega 2$

If $\Omega 1 \vdash A$, A will exhibit ***strong propositional stability*** under $\Omega 2$.

> **Proof:** Obvious. If $\Omega 1 \vdash A$, then $\Omega 2 \vdash A$. A will remain a derived tautology under $\Omega 2$ (if it was one under $\Omega 1$). It will remain true under $\Omega 2$. ■

Fact 3. Given:

> 1. Sound monotonic axiom systems $\Omega 1$, $\Omega 2$
> 2. $\Omega 1 \subset \Omega 2$
> 3. $\Gamma \vdash A$

If $\Omega 2 \vdash A$ and $\Gamma \subseteq \Omega 2$, A will exhibit:

> 1. ***Strong propositional stability*** under $\Omega 1$ only when $\Gamma \subseteq \Omega_1$
> 2. ***Propositionally instability*** otherwise.

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